(1) $\alpha-\gamma=0$, bisects angle D.
(2) $l\alpha+m\beta+n\gamma-P\beta=0$, bisects angle B.
(3) $\beta-\gamma=0$, bisects angle A.
(4) $l\alpha+m\beta+n\gamma-P\alpha=0$, bisects angle C.
(1) and (2) intersect in

$$\frac{\alpha_1}{P-m}=\frac{\beta_1}{l+n}=\frac{\gamma_1}{P-m}=\frac{2\triangle}{(a+c)(P-m)+b(n+l)}=O_2.$$

(3) and (4) intersect in

$$\frac{\alpha_2}{n+m}=\frac{\beta_2}{P-l}=\frac{\gamma_2}{P-l}=\frac{2\triangle}{a(n+m)+(b+c)(P-l)}=O_1.$$

Equation to O_1O_2 is

(5) $\alpha(P-l)+\beta(P-m)-\gamma(P+n)=0.$

The angle between (5) and $\gamma=0$ is

$$\tan\phi=\frac{\sin A-\sin D+(l\sin D-m\sin A)/P}{1+\cos A+\cos D+(n-l\cos D-m\cos A)/P}.$$

But angle $(180^\circ-F)=(A+B)$=angle BC makes with AD.
$\therefore \sin(A+B)=(l\sin D-m\sin A)/P;$
$\cos(A+B)=(n-l\cos D-m\cos A)/P.$

$$\therefore \tan\phi=\frac{\sin A-\sin D+\sin(A+B)}{1+\cos A+\cos D+\cos(A+B)}.$$

362. Proposed by V. M. SPUNAR, M. and E. E., 3536 Massachusetts Avenue, N. S., Pittsburg, Pa.

Show that the focus of an ellipse may be regarded as an indefinitely small circle having double contact with the ellipse, the directrix being the chord joining the points of contact.

Solution by PROFESSOR F. L. GRIFFIN, Williams College.

A circle with its center at $(x_0, 0)$ any point of the major axis inside the evolute $[x_0<ae^2]$, and having for its radius the length of the normal which meets the axis in that point, is tangent to the ellipse at two points, say $(x_1, \pm y_1)$. From the equation of the normal to $b^2x^2+a^2y^2=a^2b^2$ at (x_1, y_1) we find, since $a^2-b^2=a^2e^2$, $x_0=e^2x_1$; or $x_1=x_0/e^2$. Also the normal length is given by $N^2=(x_1-x_0)^2+y_1^2$, which reduces to $N^2=(1-e^2)\times(a^2-e^2x_1^2)=(1-e^2)(a^2e^2-x_0^2)/e^2$. Thus the circle has the equation

$$(x-x_0)^2+y^2=(1-e^2)(a^2e^2-x_0^2)/e^2 \qquad (1)$$

Let x increase toward the limit ae; then while the intersections are imaginary for $x_0>ae^2$, the analytical conditions for tangency are still fulfilled. For $x_0=ae$ the right member of (1) vanishes and the circle becomes the focus, since $x=x_0=ae$, $y=0$ are the only real solutions of (1). The value of x_1 has then become ae/e^2 or a/e, the abscissa of all points in the directrix.

Also solved by V. M. Spunar, G. B. M. Zerr, J. Scheffer, and Levi S. Shively.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

168. **Proposed by A. H. HOLMES, Brunswick Maine.**

Find integral values for x, y, u, and v from the following:

$uv-xy=25x+29y+29u+29v-112.$

$3v-5u+5y-x=102.$

$4y-3v=419.$

Solution by V. M. SPUNAR, M. and E. E., Pittsburg, Pa.

From (3), we have

$$v=\frac{4y-419}{3}\ldots(4),$$

which substituted in (2) and reduced will yield

$$u=\frac{9y-(x+521)}{5}\ldots(5).$$

Substituting (4) and (5) in (1), combining like terms, and we get

$$36y^2-7653y-x(19y-131)+326061=0.$$

$$\therefore x-\frac{36y^2-7653y+326061}{19y-131}=0, \text{ or } 19x-36y+\frac{140691y-6195159}{19y-131}=0, \text{ or}$$

$$361x-684y+140691-\frac{99277500}{19y-131}=0\ldots(6).$$

Hence, $99277500/(19y-131)$ must be an integer, and therefore $19y-131$ must be a factor of $99277500=2^2.3.5^4.7.31.61$. Thus

$$19y=131\pm(\alpha\,\beta\,\ldots),$$